A Hybrid LSFEM/FEM Technique for Time-Dependent Convection Dominated Equations

WCCM 2006 Los Angeles by

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Overview

- Time-Dependent Convection Dominated Equations
- Context of Convection Dominated Equations and the Navier-Stokes equations
- Different variations of the Finite Element Method FEM, SDFEM and LSFEM
- Schwarz Domain Decomposition to connect these different methods
- Numerical results
- Conclusion and Future Prospects

Time-Dependent Convection Dominated Equations

$$\begin{aligned} \frac{\partial}{\partial t}u - \varepsilon\nabla^2 u + k\cdot\nabla u &= f \text{ in } \Omega \subset \mathbb{R}^n \\ u &= h \text{ on } \partial\Omega \\ u &= g \text{ for } t = t_0 \end{aligned}$$

ε is the diffusion and $k : \mathbb{R}^n \to \mathbb{R}^n$ is the convection coefficient with $\varepsilon << \|k\|$.
Application areas :

- Heat transport
- Computational Fluid Dynamics, e.g. the Navier-Stokes equations
- ...

Context of Convection Dominated Equations and the Navier-Stokes equations

The Navier-Stokes equations :

$$\begin{aligned} \frac{\partial v}{\partial t} + (v \cdot \nabla) v - \varepsilon \nabla^2 v + \nabla p &= f \quad \text{in } \Omega && (1) \\ \nabla \cdot v &= 0 \quad \text{in } \Omega && (2) \\ v &= h \quad \text{on } \partial\Omega && (3) \\ v &= v_0 \quad \text{for } t = 0 && (4) \end{aligned}$$

Given: $\varepsilon = \frac{\eta}{\varrho}$, $\varrho =$ density, $\eta =$ viscosity, $f =$ force per unit volume
Unknowns: $v =$ velocity field, $p =$ pressure

Context of Convection Dominated Equations and the Navier-Stokes equations

Two strategies to solve the Navier-Stokes equations:

1. coupled problem

   Advantages: allows bigger time step size for high Reynolds-numbers
   divergence freedom directly guaranteed by weak formulation
   Disadvantage: huge system of equations

2. uncoupled problem using Splitting- or Projection techniques

   Advantages: system of equations is split into d+1 systems, while last pressure system is a smaller one. All equations well understood
   Disadvantage: divergence freedom not directly guaranteed by weak formulation

Context of Convection Dominated Equations and the Navier-Stokes equations

The Reynolds number

$$Re = \frac{\bar{v}l}{\varepsilon}$$

is the ratio of the characteristic length l multiplied with the mean fluid velocity and the kinematic fluid viscosity ε.

- Higher Reynolds-number requires implicit treatment of non-linear term
- This leads to a Convection Dominated Equation

Context of Convection Dominated Equations and the Navier-Stokes equations

For the Chorin-method (with $\beta_0 = 1$) and the technique developed by Heinrichs for spectral methods and extended to FEM by Frochte and Heinrichs the resulting problems for the intermediate velocity $\tilde{v} : \mathbb{R}^n \longrightarrow \mathbb{R}$ are

$$\frac{\beta_0}{\triangle t}\tilde{v}_i^{\,m+1} - \varepsilon \nabla^2 \tilde{v}_i^{\,m+1} + v^m \cdot \nabla \tilde{v}_i^{m+1} = f \qquad i = 1..d$$

- $1 \leq \beta_0 \leq 11/6$: constant depending on the used BDF scheme
- f : right side including the gradient of the pressure etc.
- v^m : velocity from the last time step

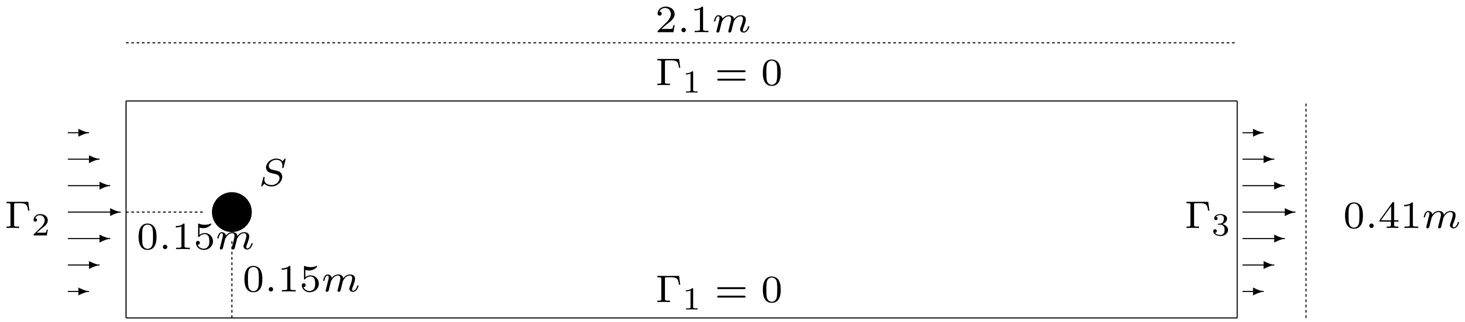
Context of Convection Dominated Equations and the Navier-Stokes equations

Flow Around a Cylinder

Defined by M. Schäfer and S. Turek in 1996

as part of

„Flow Simulation on High Performance Computers“



diameter of S $= 0.1m$, $\varepsilon = 10^{-3}$

Context of Convection Dominated Equations and the Navier-Stokes equations

$$v(0,y,t) = v(2.2,y,t) = 0.41^{-2}\sin(\pi t/8)(6y(0.41-y),0)\ , \quad 0 \le y \le 0.41\ , t \in [0,8]$$

(v)

- Behavior of v different near borders and away from them
- strong layers always at borders
- in these regions $\|v\| << 1 \Rightarrow$ diffusion is the major force in these small areas

Time-Dependent Convection Dominated Equations with FEM

Using e.g. BDF schemes we have to solve in every time step

$$
\begin{aligned}
\frac{\beta_0}{\triangle t} u - \varepsilon \nabla^2 u + k \cdot \nabla u &= Lu \\
Lu^{m+1} &= f^{m+1} - \sum_{i=1}^{n} \frac{\beta_i}{\triangle t} \tilde{u}^{m+1-i} = \tilde{f}^{m+1} \text{ in } \Omega \subset \mathbb{R}^d \\
u^{m+1} &= h \text{ on } \partial\Omega
\end{aligned}
$$

The weak formulation : Find $u^{m+1} \in V$ such that

$$(Lu^{m+1}, v) = (\tilde{f}^{m+1}, v) \qquad \forall v \in V \; .$$

Time-Dependent Convection Dominated Equations with FEM

Problems solving this equation using standard FEM are

- the stability for $\varepsilon << \|k\|$ as a result of the Cèa-Lemma

$$\|u - u_h\|_V \leq \left(\frac{C}{\varepsilon}\right)^{1/2} \inf_{v_h \in V_h} \|u - v_h\|_V \ ,$$

  with C mainly depending on k and $\frac{\beta_0}{\triangle t}$.

- the resulting system of equations. It is not symmetric and not diagonally dominant.

Time-Dependent Convection Dominated Equations with SDFEM

Find $u^{m+1} \in V$ such that

$$(Lu^{m+1}, v + \delta_T k \nabla v) = (\tilde{f}^{m+1}, v + \delta_T k \nabla v) \qquad \forall v \in V \; .$$

- SDFEM can be interpreted as Petrov-Galerkin scheme with a test-function of the type $v + \delta_T k \nabla v$
- SDFEM increases the stability adding an artificial diffusion in the flow direction
- a parameter δ has to be chosen based on the element depending parameter δ_T
- the resulting system of equations is not symmetric and not diagonally dominant

Time-Dependent Convection Dominated Equations with LSFEM

Find $u^{m+1} \in V$ such that

$$(Lu^{m+1}, Lv) = (\tilde{f}^{m+1}, Lv) \qquad \forall v \in V \ .$$

- the LSFEM formulation is stable
- no parameter has to be chosen
- the resulting system of equations is symmetric
- the formulation has a very bad condition for second order problems. So it has to be rewritten into first order problems which leads to a higher number of unknowns

Time-Dependent Convection Dominated Equations

The reduced problem

$$\begin{aligned}
\frac{\beta_0}{\triangle t}\tilde{u}^{m+1} + k \cdot \nabla \tilde{u}^{m+1} &= \tilde{f}^{m+1} \text{ in } \Omega \subset \mathbb{R}^n \\
\tilde{u}^{m+1} &= h \text{ on } \partial\Omega \\
\tilde{u}^{m+1} &= g \text{ for } t = t_0
\end{aligned}$$

It is well known - see e.g. p.176 ff “Numerical Methods for Singularly Perturbed Differential Equations” (Roos, Stynes, Tobiska) - that u beyond layers is similar to the solution $\tilde{u}$ of the first order reduced problem above.

Time-Dependent Convection Dominated Equations

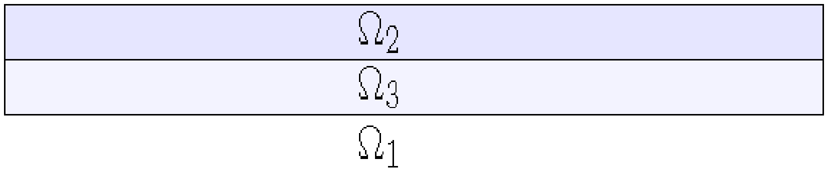
Ideas :

- Schwarz Domain Decomposition is for these problems easier, because
  1. we can use the information form the last time step
  2. the convection information flows parallel to the borders
- Decomposit Ω in three sub-domain Ω_1, Ω_2 and Ω_3
- Use FEM, SDFEM and LSFEM in the very different parts of Ω depending on their advantages and disadvantages
- Treat the less important operator - diffusion or convection - explicitly

Schwarz Domain Decomposition to connect the different Methods

- Ω_1 convection is dominant (LSFEM)
  $\frac{\beta_0}{\triangle t} v_i^{n+1} + v^n \cdot \nabla v_i^{n+1} =$
  $f^{n+1} - \sum_{i=1} \frac{\beta_i}{\triangle t} v_i^{n-i+1} + \varepsilon \nabla^2 \bar{v}_i$

- Ω_2 diffusion is dominant (FEM)
  $\frac{\beta_0}{\triangle t} v_i^{n+1} - \varepsilon \nabla^2 v_i^{n+1} =$
  $f^{n+1} - \sum_{i=1} \frac{\beta_i}{\triangle t} v_i^{n-i+1} - v^n \cdot \nabla \bar{v}_i$



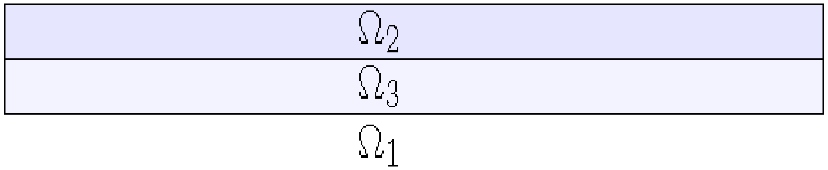
- Ω_3 smooth transition (SDFEM)
  $\frac{\beta_0}{\triangle t} v_i^{n+1} + v^n \cdot \nabla v_i^{n+1} - \varepsilon \nabla^2 v_i^{n+1}$
  $= f - \sum_{i=1} \frac{\beta_i}{\triangle t} v_i^{n-i+1}$

Schwarz Domain Decomposition to connect the different Methods

- Assume Ω_1, Ω_2 is H^2 regular $\Rightarrow$ the error based on using $\bar{v}_i$ instead of v_i tends to zero with $O(\triangle t)$.

- $\nabla^2 v$ is difficult to calculate for linear base functions in the weak formulation of LSFEM

- Because $\varepsilon << 1$, $\varepsilon\nabla^2 v$ can be disregarded receiving only a minimal error in Ω_1

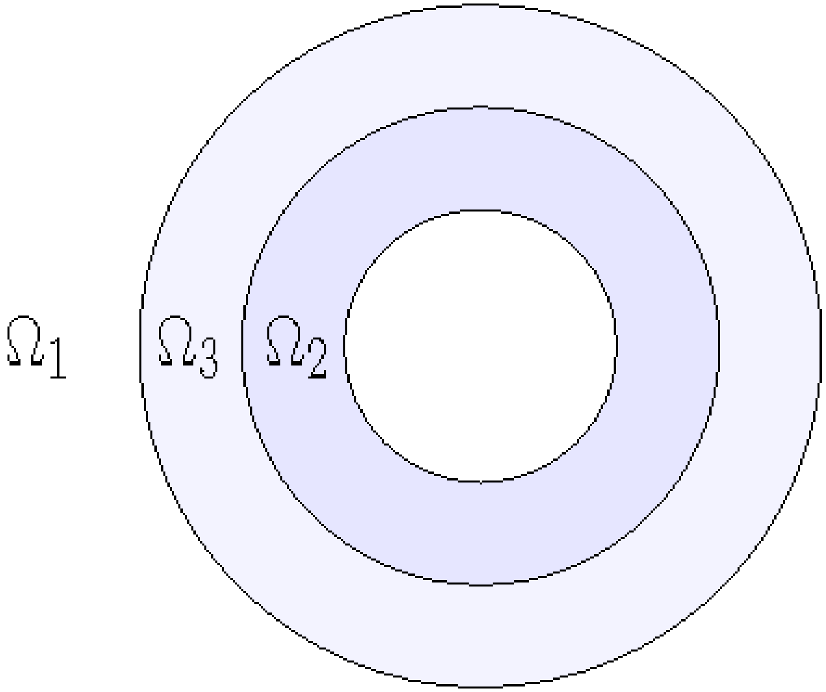


Schwarz Domain Decomposition to connect the different Methods

- Ω_1 convection is dominant (LSFEM)
  $\frac{\beta_0}{\triangle t} v_i^{n+1} + v^n \cdot \nabla v_i^{n+1} =$
  $f^{n+1} - \sum_{i=1} \frac{\beta_i}{\triangle t} v_i^{n-i+1}$

- Ω_2 diffusion is dominant (FEM)
  $\frac{\beta_0}{\triangle t} v_i^{n+1} - \varepsilon \nabla^2 v_i^{n+1} =$
  $f^{n+1} - \sum_{i=1} \frac{\beta_i}{\triangle t} v_i^{n-i+1} - v^n \cdot \nabla \bar{v}_i$

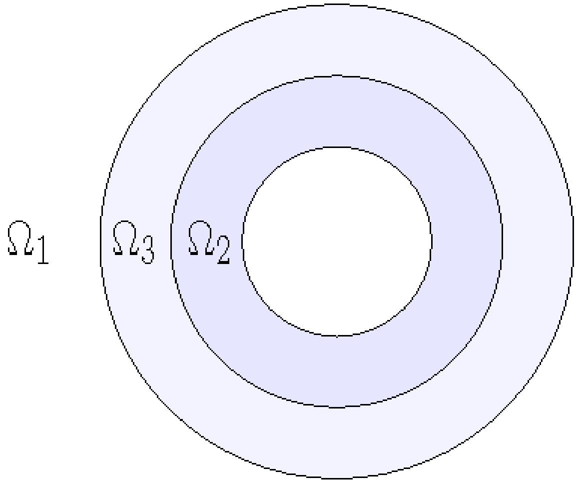
- Ω_3 smooth transition (SDFEM)
  $\frac{\beta_0}{\triangle t} v_i^{n+1} + v^n \cdot \nabla v_i^{n+1} - \varepsilon \nabla^2 v_i^{n+1}$
  $= f - \sum_{i=1} \frac{\beta_i}{\triangle t} v_i^{n-i+1}$



Schwarz Domain Decomposition to connect the different Methods

Use overlapping Schwarz Domain Decomposition to connect the different methods.

- $A_{\Omega_2} u_{\Omega_1} = b_{\Omega_1}$
  symmetric, diagonally dominant
- $A_{\Omega_1} u_{\Omega_2} = b_{\Omega_2}$
  symmetric
- $A_{\Omega_3} u_{\Omega_3} = b_{\Omega_3}$
  neither symmetric nor diagonally dominant



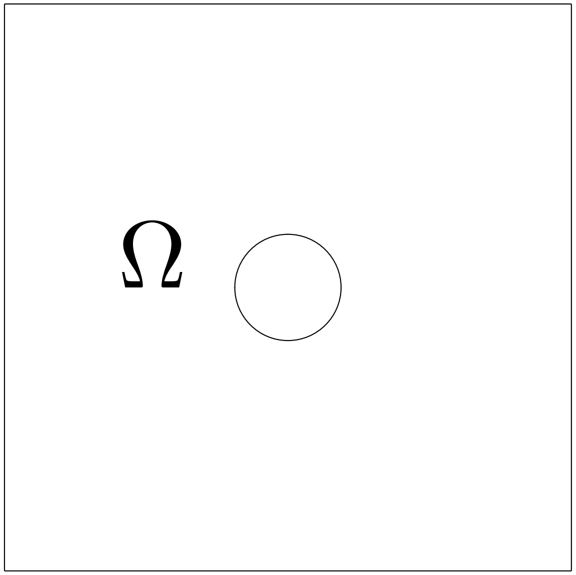
Our purpose : Keep A_{Ω_2} as small as possible

Numerical Results

The Model Problem

Ω is the square $[-1,1] \times [-1,1]$ leaving out a circle with the radius 0.25 in the middle. The triangulation is refined near the circle.

	% of the DoF	Method
$0.0625 \leq r^2 \leq 0.08$	44 %	FEM
$0.0725 \leq r^2 \leq 0.10$	+ 26 %	SDFEM
$0.0900 \leq r^2$	+ 49 %	LSFEM
	119 %	
	$\Rightarrow$ 19 %	Overlap



Numerical Results

The Model Problem

For different diffusion parameters ε we set the right f in a way that the solution is

$$u = \frac{y\cos(t)}{\sqrt{x^2+y^2}} \cdot \left(\tanh\left(\frac{0.0625-x^2+y^2}{10}\right)\right)$$

with

$$\begin{aligned} k_1 &= u(x,y,t-\triangle t) \\ k_2 &= \frac{-x\cos(t-\triangle t)}{\sqrt{x^2+y^2}} \cdot \left(\tanh\left(\frac{0.0625-x^2+y^2}{10}\right)\right) \end{aligned}$$

We use linear finite elements and the implicit Euler scheme for the time discretization.

Numerical Results

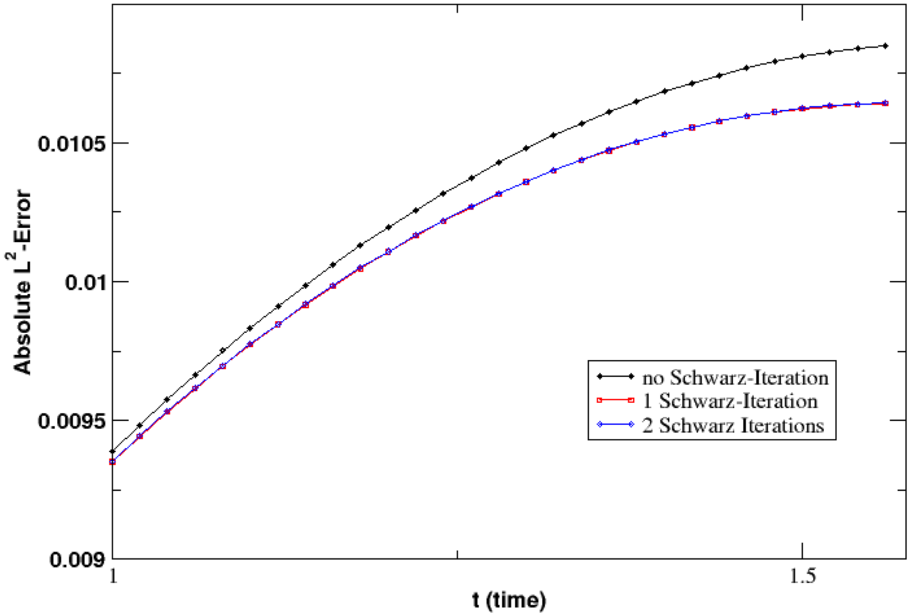
Number of Schwarz-Iterations per time step

- No Iterations
  $\Rightarrow$ no loop. The problem is solved ones per time step with the values form last time step.

- n Schwarz Iterations
  $\Rightarrow$ $n+1$ times solving the linear system of equations per time step

- For the number of Schwarz Iterations has to be chosen depending on the time discretization and how many streamlines are crossing the borders of the sub domain.

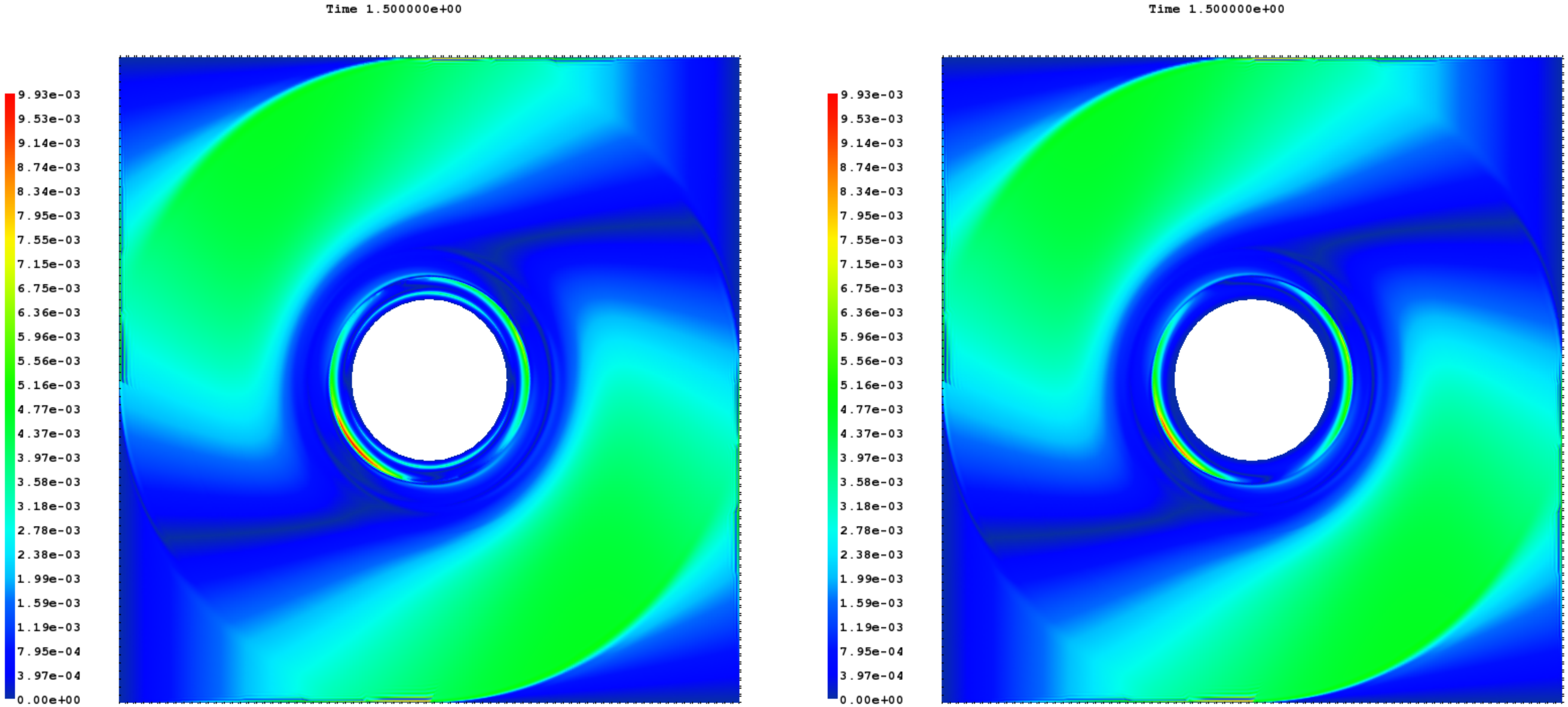
$\varepsilon = 10^-3 \; \triangle t = 1/50$



Numerical Results

$\varepsilon = 10^{-4}$

No Schwarz-Iteration **1 Schwarz-Iteration**

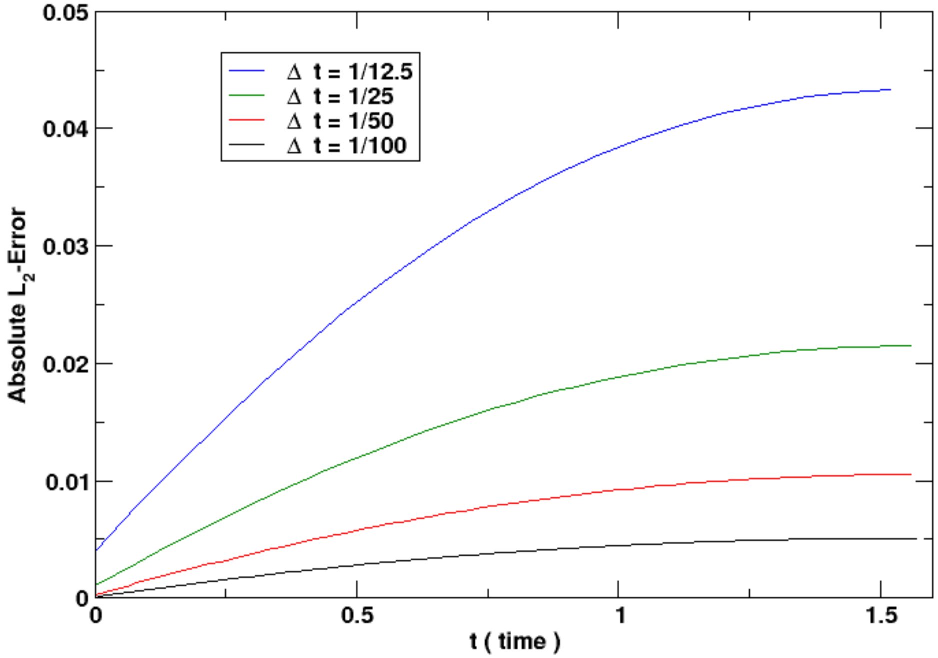


Numerical Results

Error reduction with $\triangle t$ for $\varepsilon = 10^{-4}$

- One Schwarz iteration per time step
- Results show an error reduction of first order

$\triangle t$	L_2-error	Quotient
$1/12.5$	$4.327 \cdot 10^{-2}$	-
$1/25$	$2.147 \cdot 10^{-2}$	2.01
$1/50$	$1.051 \cdot 10^{-2}$	2.04
$1/100$	$5.097 \cdot 10^{-3}$	2.06



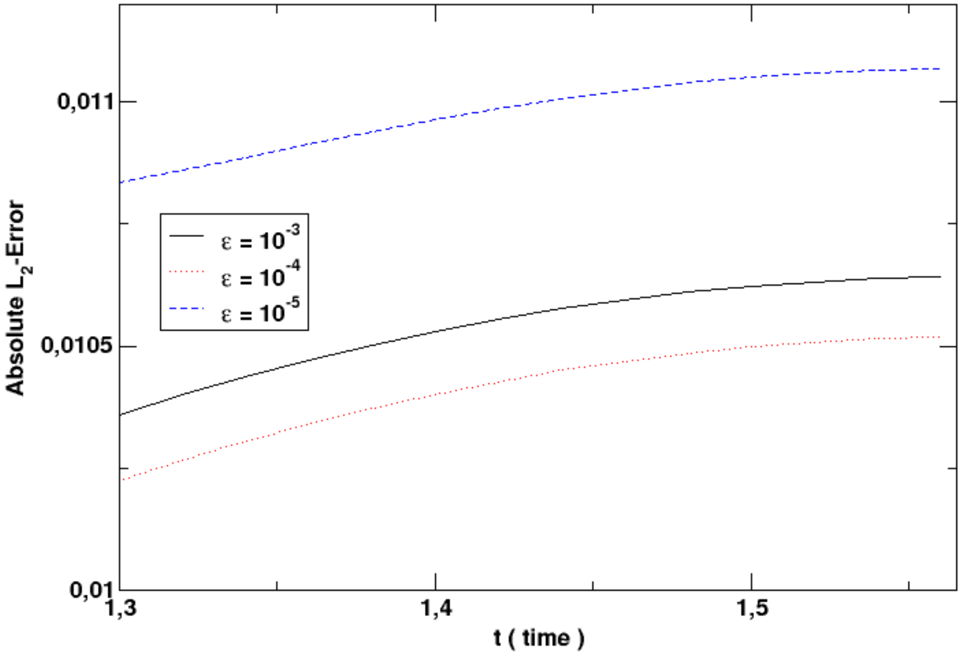
Numerical Results

Influence of ε with $\triangle t = 1/50$

- For a smaller ε the approximation in Ω_1 (LSFEM) is better

- on the other hand the finite elements in Ω_2 (FEM) and Ω_3 (SDFEM) are less accurate.

ε	L_2-error
10^{-3}	$1.064 \cdot 10^{-2}$
10^{-4}	$1.051 \cdot 10^{-2}$
10^{-5}	$1.106 \cdot 10^{-2}$



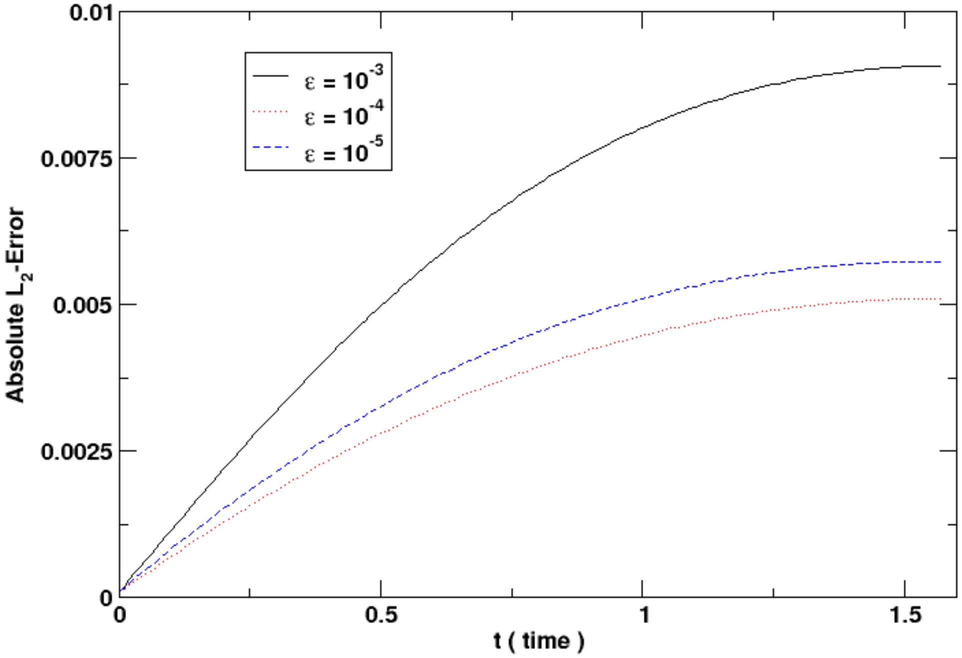
Numerical Results

Influence of ε with $\triangle t = 1/100$

- For a smaller ε the approximation in Ω_1 (LSFEM) is better

- on the other hand the finite elements in Ω_2 (FEM) and Ω_3 (SDFEM) are less accurate.

ε	L_2-error
10^{-3}	$9.066 \cdot 10^{-3}$
10^{-4}	$5.097 \cdot 10^{-3}$
10^{-5}	$5.731 \cdot 10^{-3}$



Conclusion and Future Prospects

To do :

- Use higher order base functions to evaluate the explicit treated Laplace operator
- Use BDF schemes of higher order for the algorithm
- Try an extrapolation technique for the explicit treated operators.
- Find a way to calculate the
  - size of Ω_1, Ω_2 and Ω_3
  - the number of Schwarz iterations per time step

Conclusion and Future Prospects

We can sum up that the presented technique can

- split the huge problem using overlapping Schwarz in three smaller problems
- two of the three problems can be solve using very effective methods like e.g. Multgrids.
- For the third problem form Ω_3 we have $\|k\|_{\Omega_3} \leq \|k\|_{\Omega}$ and so it is *less unsymmetric* and we can manage it with less streamline diffusion.

and so has the potential to make splitting methods for the Navier-Stokes-Equation much more effective.

END

Thanks a lot for your

attention